Path Integral Representations for the Spin-Pinned Quantum XXZ Chain

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Abstract

Two discrete path integral formulations for the ground state of a spin- pinned quantum anisotropic XXZ Heisenberg chain are introduced. Their properties are discussed and two recursion relations are proved.

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1 Introduction

We introduce in this work a path integral representation for the ground state of the anisotropic Heisenberg XXZ model with a pinned-spin as a suitable random walk on two dimensional lattices. Our representation generalises what we had previously introduced for the standard XXZ chain. The reason to introduce spin-pinned models is to deal with localised impurities in magnetic materials [5, 6, 7]. The path integral representation is of great help in establishing the properties of the model under investigation: the square norm of the quantum state vector admits the interpretation of the path integral partition function, and the probabilistic features, in particular related to Markov type properties, play a decisive role in evaluating correlation functions and other physical quantities. We prove here two recursion relations that express the properties of systems of a given size in terms of those of smaller size. Recursion relations of this kind have been used successfully to derive bounds on correlation functions in [4].

2 Path Integral Models in $\mathbb{Z}^2$

Let us consider the two dimensional lattice $\mathbf{Z}^2$. A "zig-zag" path p is a connected path of unit steps in $\mathbf{Z}^2$ monotonically increasing in both coordinates. For each $t = 1, 2, 3...$ the path will be encoded in a sequence of $\alpha(t) \in \{0, 1\}$ conventionally associating $\alpha = 1$ to a horizontal step and $\alpha = 0$ to a vertical one. We denote by $|p|$ the length of the path i.e. the sum of the steps (see Figure 1). A path integral model on $\mathbf{Z^2}$ is a law that associates positive weights $w(p)$ to a given set of paths.

Let $\mathcal{P}_{I,F}$ denote the set of all paths from an initial point $I = (n_0, m_0)$ to a final one $F(n_f, m_f)$ for $n_0 \leq n_f$ and $m_0 \leq m_f$. The collection of all such paths visits all the points of the *rectangle* $[I, F]$. The *canonical* partition function is defined as

$$Z(I, F) \;=\; \sum_{p \in \mathcal{P}_{I,F}} w(p) \,, \tag{2.1}$$

which induce the probability measure on $\mathcal{P}_{I,F}$ given by

$$\mathrm{Prob}(p) \;=\; \frac{w(p)}{Z(I, F)} \,. \tag{2.2}$$

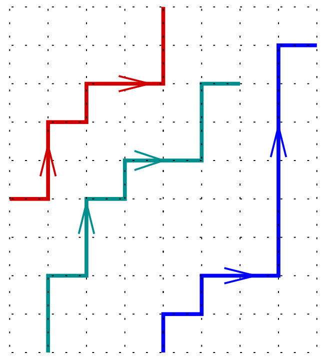

Figure 1: Examples of zig-zag paths on $\mathbf{Z^2}$.

In path integral models, correlation functions measure the probability that a path goes through particular points $Q_1 = (i_1, j_1), Q_2 = (i_2, j_2), \cdots, Q_r = (i_r, j_r)$, with

$$n_o \leq i_1 \leq i_2 \leq ... \leq n_f \ , \tag{2.3}$$

and

$$m_o \leq j_1 \leq y_2 \leq ... \leq m_f \ . \tag{2.4}$$

The one-point correlation function is defined as the probability of crossing a single point Q

$$P_{I,F}(Q) = \frac{Z(I,F \mid Q)}{Z(I,F)} \ , \tag{2.5}$$

where

$$Z(I,F \mid Q) \ = \sum_{p \in \mathcal{P}_{I,F}(Q)} w(p) \tag{2.6}$$

and $\mathcal{P}_{(I,F)}(Q)$ is the set of paths from the I to F that pass through the point Q. More generally, we can define

$$P_{I,F}(Q_1; \cdots; Q_r) = \frac{Z(I,F \mid Q_1; \cdots; Q_r)}{Z(I,F)}, \tag{2.7}$$

where

$$Z(I,F \mid Q_1; \cdots; Q_r) \ = \sum_{p \in \mathcal{P}_{I,F}(Q_1; \cdots; Q_r)} w(p) \tag{2.8}$$

and $\mathcal{P}_{I,F}(Q_1; \cdots; Q_r)$ denotes the set of paths that pass through the particular points $Q_1, Q_2, \cdots, Q_r$.

In this framework, we consider models for which the weight $w(p)$ is a local function of the bonds that the path is passing through. Denoting by $\mathbf{B}^2$ the set of bonds in $\mathbf{Z}^2$, we associate a positive number $w(b)$ to each element b of $\mathbf{B}^2$ and define

$$w(p) = \prod_{b \in p} w(b). \tag{2.9}$$

More generally for a given finite set of paths $\mathcal{P}$ (the *ensemble*) we define

$$\mathcal{Z} \;=\; \sum_{p \in \mathcal{P}} w(p) \;, \tag{2.10}$$

for a set of paths $\mathcal{P}(Q)$ through a point Q,

$$\mathcal{Z}(Q) \;=\; \sum_{p \in \mathcal{P}(Q)} w(p) \tag{2.11}$$

and for a set of paths $\mathcal{P}^{(+)}(Q)$ (resp. $\mathcal{P}^{(-)}(Q)$) *ending* (resp. *beginning*) in Q,

$$\mathcal{Z}^{(\pm)}(Q) \;=\; \sum_{p \in \mathcal{P}^{(\pm)}(Q)} w(p) \;. \tag{2.12}$$

In order to prove the following basic *Markov* property we show the following lemma.

Lemma 2.1 (Markov property)

$$Z(I, F \mid Q_1; \cdots; Q_r) \;=\; Z(I, Q_1) Z(Q_1, Q_2) \cdots Z(Q_r, F) \tag{2.13}$$

and analogously

$$\mathcal{Z}(Q) \;=\; \mathcal{Z}^{(+)}(Q) \mathcal{Z}^{(-)}(Q) \tag{2.14}$$

Proof. These identities follow from the fact that the paths are increasing in both coordinates and from (2.9).

■

Corollary 2.1 *Let the set $\mathcal{S}_I(l)$ (sphere of center I and radius l) be the points reachable by the paths p starting at I of lenght l. For any sphere $\mathcal{S}_I(l)$ such that $l \leq n_f + m_f - n_0 - m_0$ we have*

$$Z(I, F) \;=\; \sum_{Q \in \mathcal{S}_I(l)} Z(I, Q) Z(Q, F) \tag{2.15}$$

Proof. We write (2.1) with the set of paths $\mathcal{P}_{(I,F)} = \cup_{Q\in\mathcal{S}_I(l)}\mathcal{P}_{I,F}(Q)$ as

$$Z(I,F) \;=\; \sum_{\cup_{Q\in\mathcal{S}_I(l)}\mathcal{P}_{I,F}(Q)}\;\sum_{p\in\mathcal{P}_{I,F}(Q)} w(p). \tag{2.16}$$

and by lemma (2.1) we have the corollary.

■

In a completely analogous way the following can be proved:

Corollary 2.2 *Let b_h and b_v the two bonds leading to (resp. departing from) Q with $b_h = (Q_h, Q)$ and $b_v = (Q_v, Q)$. Then*

$$\mathcal{Z}^{(\pm)}(Q) \;=\; w(b_h)\mathcal{Z}^{(\pm)}(Q_h) + w(b_v)\mathcal{Z}^{(\pm)}(Q_v) \tag{2.17}$$

3 The XXZ Spin-Pinned Chain

In one dimension, for $0 < q < 1$, we consider the Hamiltonian

$$H_{[-L,K]} = \sum_{x=-L}^{-1} h_x^{(q^{-1})} + \sum_{x=0}^{K-1} h_x^{(q)}, \tag{3.18}$$

where

$$h_x^{(q)} = -\frac{2}{q+q^{-1}}(S_x^{(1)}S_{x+1}^{(1)} + S_x^{(2)}S_{x+1}^{(2)}) - (S_x^{(3)}S_{x+1}^{(3)} - 1/4) - \frac{q^{-1}-q}{2(q^{-1}+q)}(S_x^{(3)} - S_{x+1}^{(3)}) \tag{3.19}$$

is the orthogonal projection on the vector

$$\xi_q = \frac{1}{\sqrt{1+q^2}}(q\,|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \tag{3.20}$$

and $S_x^{(i)}$ $(i = 1,2,3)$ are the usual Pauli spin matrices at the site x. From the definition of ξ_q if follows that

$$h^q\,|\downarrow\downarrow\rangle = 0, \qquad h^q\,|\downarrow\uparrow\rangle = \frac{1}{q+q^{-1}}\,(q\,|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)\,, \tag{3.21}$$

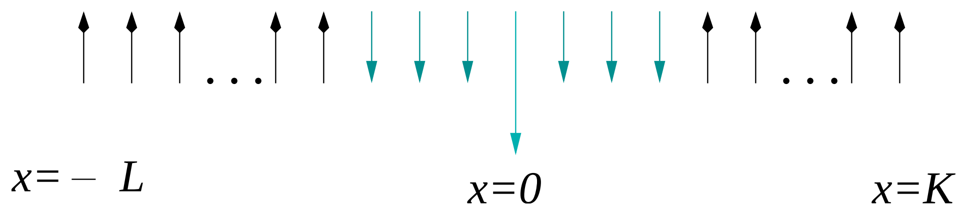

Figure 2: The spin-pinned chain.

$$h^q \left|\uparrow\uparrow\right\rangle = 0, \qquad h^q \left|\uparrow\downarrow\right\rangle = -\frac{1}{q+q^{-1}}\left(\left|\downarrow\uparrow\right\rangle - q^{-1}\left|\uparrow\downarrow\right\rangle\right). \tag{3.22}$$

The Hamiltonian (3.18) represents a spin-1/2 XXZ ferromagnetic chain (see [1, 2]) of lenght $K+L+1$ with a kink-antikink structure: two $(+)$ boundary fields and a spin pinned at the origin through the action of a local $(-)$ field whose strenght is twice the one on the boundaries (see Figure 2). The configuration of spins in the one-dimensional chain is identified with the set of numbers α_x for $x = \{-L, ..., K\}$ where α takes values in $\{0,1\}$. We choose $\alpha = 0$ to correspond to an up spin, or, in the particle language, to an unoccupied site. Conversely, $\alpha = 1$ corresponds to a down spin or an occupied site. We will be interested in the ground state of the model in the sector with N down spins. We have the following result.

Theorem 1 *A ground state of the model is given by*

$$\psi_N(-L,K) = \sum_{\{\alpha_x\}\in\mathcal{A}_{N,M}} \phi(\{\alpha_x\}) \mid \{\alpha_x\}\rangle \tag{3.23}$$

where the $\mathcal{A}_{N,M}$ the set of configurations $\{\alpha_x\}$ such that $\sum_x \alpha_x = N$ with the condition $N+M = L+K+1$, and the functions $\phi(\alpha)$ satisfy the set of equations

$$\begin{aligned} \phi(...,\ \sigma_x = \uparrow,\ \sigma_{x+1} = \downarrow,\ ...) &= q^{-1}\phi(...,\ \sigma_x = \downarrow,\ \sigma_{x+1} = \uparrow,\ ...) \quad \text{for} \quad x < 0, \\ \phi(...,\ \sigma_x = \uparrow,\ \sigma_{x+1} = \downarrow,\ ...) &= q\ \phi(...,\ \sigma_x = \downarrow,\ \sigma_{x+1} = \uparrow,\ ...) \quad \text{for} \quad x \geq 0. \end{aligned} \tag{3.24}$$

Proof. Follows by direct substitution of (3.23) in (3.18) using (3.21) and (3.22).

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Theorem 2 *The function*

$$\phi(\alpha) = \prod_{x=-L}^{K} q^{|x|\alpha_x} \tag{3.25}$$

is the solution of (3.24).

Proof. Since the set of equations (3.24) admits a unique solution in each sector of fixed down spins, we are left with proving that (3.25) satisfies

$$\frac{\phi(\cdots,\ \alpha_x = 1,\ \alpha_{x+1} = 0,\ \cdots)}{\phi(\cdots,\ \alpha_x = 0,\ \alpha_{x+1} = 1,\ \cdots)} = \begin{cases} q & \text{when } x < 0, \\ q^{-1} & \text{when } x \geq 0 \end{cases} \tag{3.26}$$

Since (3.26) equals

$$\frac{q^{|x|}}{q^{|x+1|}}, \tag{3.27}$$

the proof is complete.

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The norm of the ground state vector (3.23) with n spins down is

$$\|\psi_N(-L, K)\|^2 = \sum_{\{\alpha_x\} \in \mathcal{A}_{N,M}} \prod_{x=-L}^{K} q^{2|x|\alpha_x}. \tag{3.28}$$

4 Two Path Integral Representations.

The two path integral representations that we introduce here are based on the one introduced in [4] for the anisotropic XXZ quantum chain (with no pinning field), which we first recall here for completeness.

Theorem 3 (Path integral representation for interface ground state [4])

$$Z_q(n, m) \ = \sum_{\{\alpha_x\} \in \mathcal{A}_{n,m}} \prod_{x=1}^{K} q^{2x\alpha_x} \ = \sum_{p \in \mathcal{P}_{(n,m)}} w(p) \tag{4.29}$$

is the partition function for the classical path integral model associated with the quantum XXZ model with n down spins and m up spins ($n + m = K$) for the following choice of weights

$$w(b) = \begin{cases} q^{2(i_b + j_b)} & \text{for a horizontal bond whose right end is at } (i_b, j_b) \\ 1 & \text{any vertical bond .} \end{cases} \tag{4.30}$$

The partition function (4.29) has the following explicit expression:

$$Z_q(n,m) \ = \ q^{n(n+1)} \ \frac{\prod_{i=1}^{n+m}(1-q^{2i})}{\prod_{i=1}^{n}(1-q^{2i}) \ \prod_{i=1}^{m}(1-q^{2i})}. \tag{4.31}$$

Moreover for every $I = (n',m')$, $F = (n,m)$ and $P = (x,y)$ with $x \le n' \le n$ and $y \le m' \le m$

$$Z_q(I,F) \ = \ q^{2(x+y)(n-n')} Z_q(I',F'), \tag{4.32}$$

where $I' = (n'-x, m'-y)$, and $F' = (n-x, m-y)$.

Next, we state the definitions of two path integral representations, i.e., two path measures. Although the measures are different, it will turn out that both measures generate the same family of partition functions.

Definition 4.1 (Path Integral representation 1) *To each configuration of $\alpha \in \mathcal{A}_{N,M}$ representing a spin configuration for the chain in $[-L,K]$ we associate a path $p(\alpha)$ starting from the origin of the lattice and ending at N,M described by the sequence of $\alpha(t)$, $1 \le t \le L+K+1$*

$$\alpha(t) = \begin{cases} \alpha_t & \text{for} \quad 1 \le t \le K \\ \alpha_{t-L-K-1} & \text{for} \quad K \le t \le K+L+1 \,, \end{cases} \tag{4.33}$$

and consider the weights system defined by

$$w(b) = \begin{cases} q^{2(i_b+j_b)} & \text{for} \quad i_b+j_b \le K \\ q^{2(K+L+1)-2(i_b+j_b)} & \text{for} \quad K \le i_b+j_b \le K+L+1 \\ 1 & \text{for} \quad \text{any vertical bond.} \end{cases} \tag{4.34}$$

We will denote by $Z(N,M)$ the partition function corresponding to the given weights and the set of paths $\mathcal{P}_{(0,0),(N,M)}$. See Figure 3.

Definition 4.2 (Path Integral representation 2) *In this case the spin configuration will correspond to a path*

$$\alpha(t) \ = \ \alpha_{t-L-1} \tag{4.35}$$

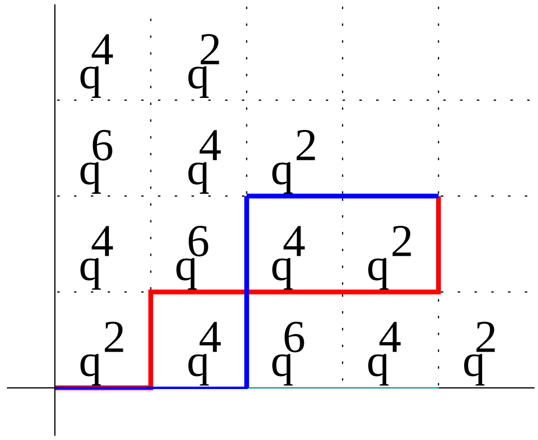

Figure 3: The first path representation showing two paths. The weights (4.34) are indicated on the bonds. Note that the weights are constant along lattice spheres of center $(0,0)$ and radius l for $1 \leq l \leq N+M$.

and consider the weights system defined by

$$w(b) = \begin{cases} q^{2|i_b+j_b|} & \text{for} \quad \text{horizontal bonds} \\ 1 & \text{for} \quad \text{vertical bonds.} \end{cases} \tag{4.36}$$

The set of paths $\widetilde{\mathcal{P}}_0(N,M)$ is the set containing all paths departing from the third quadrant sphere of radius $L+1$, and arriving at the first quadrant sphere of radius K, with a total number of horizontal bonds equal to N and passing through the origin of the lattice. The corresponding partition function is then

$$\mathcal{Z}(N,M) \;=\; \sum_{p\in\widetilde{\mathcal{P}}_0(N,M)} w(p)\ . \tag{4.37}$$

See Figure 4.

Theorem 4

$$\|\psi_N(-L,K)\|^2 \;=\; Z(N,M) \;=\; \mathcal{Z}(N,M) \tag{4.38}$$

Proof. The first equality comes from Theorem 3, (4.33) and (4.34). The second equality is again a consequence of Theorem 3, (4.35) and (4.36).

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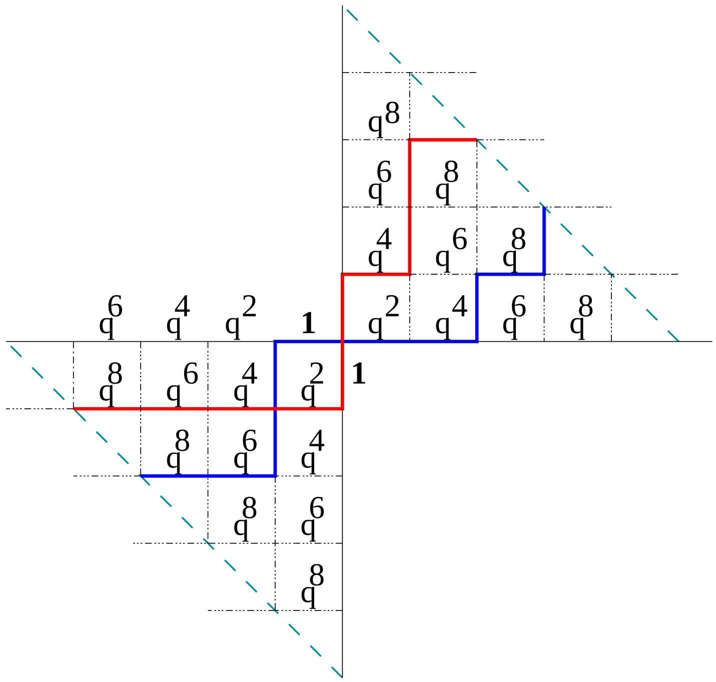

Figure 4: The second path representation.

5 Recursion Relations.

The partition functions $\mathcal{Z}(N,M)$, defined in (4.37), can be related to explicitly known objects such as the $Z_q(n,m)$ given in (4.31). This can be used effectively in numerical or symbolic computations.

Theorem 5 *The partition function (4.37) fulfills the following relation*

$$\mathcal{Z}(N,M) = \sum_{n+n'=N} Z_q(n, K-n) \cdot \{Z_q(n'-1, L-n'+1) + Z_q(n', L-n')\} \ . \qquad (5.39)$$

Proof. Consequence of (2.17).

∎

This expression for the partition function of the model can be written in terms of the partition function of a genuine anisotropic Heisenberg model for $1 \leq x \leq N+M$, as we now show. Our first results is:

Theorem 6 *The partition function (5.39) is given by*

$$\mathcal{Z}(N,M) = Z_q(N,M)\ \langle q^{-2(K+1)S_L}\rangle_{N,M} \qquad (5.40)$$

where $Z_q(N,M)$ is the partition function of the anisotropic Heisenberg model, $S_L = \sum_{x=-L}^{0} \alpha_x$ and the symbol $\langle\cdot\rangle$ denotes the expectation value in the canonical ensemble of the anisotropic Heisenberg model.

Proof. We apply the property (4.32) to translate the partition functions in (5.39):

$$Z_q(n'-1, L+1-n') = q^{-2(K+1)(n'-1)} Z_q(N-n'+1, M-L-1+n'; N, M), \quad (5.41)$$

In the same way we obtain

$$Z_q(n', L-n') = q^{-2(K+1)n'} Z_q(N-n', M-L+n'; N, M) \quad (5.42)$$

for the second term in (5.39). Substituting (5.41) and (5.42) into (5.39) we get

$$\begin{aligned} \mathcal{Z}(N,M) &= \sum_{n+n'=N} q^{-2(K+1)n'} Z_q(n, K-n) \{ Z_q(N-n', M-L+n'; N, M) \\ &+ q^{2(K+1)} Z_q(N-n'+1, M-L-1+n'; N, M) \} . \end{aligned} \quad (5.43)$$

By using the Theorem 4.32, we rewrite the terms between braces as follows

$$\mathcal{Z}(N,M) = \sum_{n+n'=N} q^{-2(K+1)n'} Z_q(n, K-n) Z_q(n, K-n; N, M). \quad (5.44)$$

The above expression can be interpreted as the average value of $q^{-2(K+1)(N-n)}$ over all the paths from the origin to (N,M) that pass through the point (n,m). Here, n is the number of down spins, i.e., horizontal steps in the path, to the left of the pinning site. As the quantity to be averaged only depends n, not on the individual path, we just need to know the distribution of n, which is given by the probabilities $p_{N,M}(n)$, defined by

$$p_{N,M}(n) = \frac{Z_q(n, n-N) Z_q(n, n-N; N, M)}{\sum_n Z_q(n, n-N) Z_q(n, n-N; N, M)} = \frac{Z_q(n, n-N) Z_q(n, n-N; N, M)}{Z_q(N,M)}. \quad (5.45)$$

Then the result (5.40) follows and this completes the proof of the theorem.

■

Theorem 7 *The partition function satisfies*

$$\mathcal{Z}(N,M) = \mathcal{Z}(N-1,M) + \mathcal{Z}(N,M-1) \quad (5.46)$$

Proof. Consequence of (4.33) and (2.15).

■

Theorem 8 *The partition function satisfies*

$$\mathcal{Z}(N,M) \;=\; \sum_{n+m=K} Z_q(n,m)[Z_q(N-n,M-m-1)+Z_q(N-n-1,M-m)] \quad (5.47)$$

Proof. Consequence of (2.15) and of the observation

$$Z(n,m;N,M-1) \;=\; Z_q(N-n,M-m-1) \quad (5.48)$$

and

$$Z(n,m;N-1,M) \;=\; Z_q(N-n-1,M-m) \quad (5.49)$$

which can be derived from (4.33).

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